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Integrable couplings of relativistic Toda lattice systems in polynomial form and rational form, their hierarchies and bi-Hamiltonian structures

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Abstract
Integrable couplings of relativistic Toda lattice systems in polynomial form and rational form, and their hierarchies, are derived from a four-by-four discrete matrix eigenvalue problem. The bi-Hamiltonian structure for every integrable coupling in the two hierarchies obtained is established by means of the discrete variational identity. Ultimately, Liouvolle integrability of the obtained integrable couplings is demonstrated.

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1. Introduction

In recent years, integrable lattice systems, treated as models of many physics phenomena, have received considerable attention. Many integrable nonlinear lattice systems have been derived. Their various algebraic and geometric properties have been studied from different points of view [1–17]. In the lattice soliton theory, it is still an important and complicated task to search for new integrable nonlinear lattice systems. Discrete zero curvature representation is an effective method of constructing integrable nonlinear lattice systems.

A hierarchy of lattice systems

$$u_{nt_m} = K_m(u_n, Eu_n, E^{-1}u_n, \ldots), \qquad m \geqslant 0, \tag{1}$$

is called Lax integrable if it can be rewritten as a compatibility condition

$$U_{nt_m} = \left(EV_n^{(m)}\right)U_n - U_nV_n^{(m)}, \qquad m \geqslant 0, \tag{2}$$

of a discrete spatial eigenvalue problem

$$E\varphi_n = U_n\varphi_n = U_n(u_n, \lambda)\varphi_n \tag{3}$$

and a sequence of appropriate temporal eigenvalue problems

$$\varphi_{nt_m} = V_n^{(m)}(u_n, Eu_n, E^{-1}u_n, \ldots)\varphi_n, \qquad m \geqslant 0, \tag{4}$$

where λ is an eigenvalue and $\lambda_t = 0$. The shift operator E and the inverse of E are defined by

$$Ef_n = f_{n+1}, \qquad E^{-1}f_n = f_{n-1}, \qquad n \in Z. \tag{5}$$

Equations (3) and (4) are said to be a Lax pair of the hierarchy of integrable lattice systems (1). Usually, the U_n and $V_n^{(m)}$ are 2×2 matrices.

It is well known that the relativistic Toda lattice models

$$\begin{cases} r_{nt} = r_n(s_{n-1} - s_n) + r_n(r_{n-1} - r_{n+1}), \\ s_{nt} = r_n s_n - r_{n+1} s_n, \end{cases} \tag{6}$$

and

$$\begin{cases} r_{nt} = \dfrac{r_n}{s_{n-1}} - \dfrac{r_n}{s_n}, \\ s_{nt} = \dfrac{r_{n+1}}{s_{n+1}} - \dfrac{r_n}{s_{n-1}}, \end{cases} \tag{7}$$

are famous discrete integrable systems. They have important applications in physics, and have been widely discussed [4–9]. In equations (6) and (7), $r_n = r(n,t)$ and $s_n = s(n,t)$ are real functions defined over $Z \times R$. Equation (6) is called the relativistic Toda lattice system in polynomial form, and equation (7) is called the relativistic Toda lattice system in rational form. The relativistic Toda lattice systems (6) and (7) are a source of new integrable lattice systems; many integrable lattice systems related to the relativistic Toda lattice equations (6) or (7) have been proposed and studied [12, 13].

There are many methods of generating new integrable lattice systems, one of which is used to extend the known integrable lattice systems to larger and complicated ones from the points of view of both potentials and dimensions. Recently, more and more attention has been paid to the investigation of integrable couplings of soliton equations, both in the continuous and discrete cases [18–28]. A few methods of constructing integrable couplings of the known integrable systems are presented using the perturbation method [18, 19], enlarging eigenvalue problems [20], semi-direct sums of Lie algebras [21–24] and so forth.

For a given hierarchy of integrable lattice systems,

$$u_{nt_m} = K_m(u_n, Eu_n, E^{-1}u_n, \ldots), \qquad m \geqslant 0,$$

in which $u_n = u(n,t)$ is commonly a vector-valued real function defined over $Z \times R$. We actually want to construct a new bigger triangular hierarchy of integrable lattice systems as follows:

$$\begin{pmatrix} u_n \\ z_n \end{pmatrix}_{t_m} = \begin{pmatrix} K_m(u_n) \\ S_m(u_n, z_n) \end{pmatrix}. \tag{8}$$

In equation (8), z_n is a new vector-valued real function defined over $Z \times R$, and the vector-valued function $S_m(x_n, y_n)$ should satisfy the non-triviality condition $\frac{\partial S_n(u_n,z_n)}{\partial v} \neq 0$, in which $v = u_n, Eu_n, E^{-1}u_n, \ldots$. In addition, an important task in the theory of integrable lattice systems is to establish the Hamiltonian structures for the integrable couplings under consideration. Previously, the Hamiltonian structures of integrable lattice systems may have been established by the discrete trace identity [6]. But it cannot be used to establish the Hamiltonian structures of discrete integrable couplings [24]. In order to establish the Hamiltonian structures of discrete integrable couplings, in [24], the discrete trace identity was generalized to a discrete variational identity in the matrix Lie algebra which possesses a non-degenerate symmetric bilinear form. Using the method of discrete zero curvature representation, the relativistic Toda lattice equations (6) and (7) may be derived from the same discrete matrix eigenvalue problem:

$$E\varphi_n = Y_n\varphi_n, \qquad Y_n = \begin{pmatrix} 0 & 1 \\ r_n & \lambda + \frac{s_n}{\lambda} \end{pmatrix}, \qquad \varphi_n = \begin{pmatrix} \varphi_n^1 \\ \varphi_n^2 \end{pmatrix}. \tag{9}$$

In this paper, we would like to introduce the 4×4 discrete matrix eigenvalue problem

$$E\psi_n = U_n(u_n, \lambda)\psi_n, \qquad U_n(u_n, \lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ r_n & \lambda + \frac{s_n}{\lambda} & v_n & \frac{w_n}{\lambda} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & r_n & \lambda + \frac{s_n}{\lambda} \end{pmatrix}, \tag{10}$$

in which $u_n = (r_n, s_n, v_n, w_n)^T$, $r_n = r(n,t)$, $s_n = s(n,t)$, $v_n = v(n,t)$, $w_n = w(n,t)$ are real functions defined over $Z \times R$ and $\psi_n = \left(\psi_n^1, \psi_n^2, \psi_n^3, \psi_n^4\right)^T$ is the eigenfunction vector. Let us set

$$Z_n = \begin{pmatrix} 0 & 0 \\ v_n & \frac{w_n}{\lambda} \end{pmatrix}.$$

It is evident that the

$$U_n = \begin{pmatrix} Y_n & Z_n \\ \tilde{0} & Y_n \end{pmatrix},$$

where $\tilde{0}$ is the 2×2 zero matrix. Therefore, equation (10) is an enlarging eigenvalue problem of equation (9).

This paper is organized as follows. In section 2, starting from equation (10), we first briefly deduce the relativistic Toda lattice hierarchies, which include the polynomial form and rational form. In section 3, by dint of the appropriate temporal eigenvalue problems (4), in which $V_n^{(m)}$, $m \geqslant 0$, only have positive powers of the eigenvalue λ, we derived the positive integrable coupling hierarchy associated with enlarging the relativistic Toda lattice eigenvalue problem (10). In the obtained positive integrable coupling hierarchy, a typical member is the following integrable coupling of equation (6):

$$\begin{cases} r_{nt_1} = r_n(r_{n-1} - r_{n+1}) + r_n(s_{n-1} - s_n), \\ s_{nt_1} = s_n(r_n - r_{n+1}), \\ v_{nt_1} = r_n(s_{n-1} - s_n + w_{n-1} - w_n + r_{n-1} - r_{n+1} + v_{n-1} - v_{n+1}) \\ \qquad + v_n(r_{n-1} - r_{n+1} + s_{n-1} - s_n), \\ w_{nt_1} = s_n(r_n + v_n - r_{n+1} - v_{n+1}) + w_n(r_n - r_{n+1}). \end{cases} \tag{11}$$

If we set $v_n = -r_n$, $w_n = -s_n$, then equation (11) becomes

$$\begin{cases} r_{nt} = r_n(r_{n-1} - r_{n+1}) + r_n(s_{n-1} - s_n), \\ s_{nt} = s_n(r_n - r_{n+1}). \end{cases}$$

This is just the relativistic Toda lattice model in polynomial form (6). In section 3, we construct the corresponding negative integrable coupling hierarchy from other temporal eigenvalue problems (4), in which $\widehat{V}_n^{(m)}$, $m \geqslant 0$, only have negative powers of the eigenvalue λ. In the negative integrable coupling hierarchy obtained, the representative member is the integrable coupling of equation (7):

$$\begin{cases} r_{nt_1} = \frac{r_n}{s_{n-1}} - \frac{r_n}{s_n}, \\ s_{nt_1} = \frac{r_{n+1}}{s_{n+1}} - \frac{r_n}{s_{n-1}}, \\ v_{nt_1} = \frac{r_n}{s_{n-1}} - \frac{r_n}{s_n} + r_n\left(\frac{w_n}{s_n^2} - \frac{w_{n-1}}{s_{n-1}^2}\right), \\ w_{nt_1} = \frac{r_{n+1}}{s_{n+1}} - \frac{r_n}{s_{n-1}} + \frac{v_{n+1}}{s_{n+1}} - \frac{v_n}{s_{n-1}} + \frac{r_n w_{n-1}}{s_{n-1}^2} - \frac{r_{n+1} w_{n+1}}{s_{n+1}^2}. \end{cases} \tag{12}$$

If we set $v_n = \frac{1}{3}r_n, \quad w_n = \frac{2}{3}s_n$, equation (12) is then reduced to the relativistic Toda lattice equation in the rational form

$$\begin{cases} r_{nt} = \frac{r_n}{s_{n-1}} - \frac{r_n}{s_n}, \\ s_{nt} = \frac{r_{n+1}}{s_{n+1}} - \frac{r_n}{s_{n-1}}. \end{cases}$$

In section 5, we establish the Hamiltonian structures and construct the discrete bi-Hamiltonian formulation for the positive and negative discrete integrable couplings obtained, by means of the discrete variational identity, which is constructed through a non-degenerate symmetric bilinear form. Then, it is shown that the discrete Hamiltonian systems obtained have an infinite sequence commuting conserved functionals. Therefore, they are all Liouville integrable. Finally, in section 6, there are some conclusions and remarks.

2. The relativistic Toda lattice hierarchies

In this section, we derive briefly the relativistic Toda lattice hierarchies, both the polynomial form and rational form, from the same matrix eigenvalue problem (9) [10]. From the stationary discrete zero-curvature equation

$$(EY_n)\Pi_n - \Pi_n Y_n = 0, \tag{13}$$

with

$$\Pi_n = \sum_{m=0}^{\infty} \begin{pmatrix} a_n^{(m)}\lambda & b_n^{(m)} \\ c_n^{(m)} & -a_n^{(m)}\lambda \end{pmatrix} \lambda^{2m-1},$$

we obtain the initial relation

$$\left(a_{n+1}^{(0)} - a_n^{(0)}\right) = c_{n+1}^{(0)} - rb_n^{(0)}, \qquad b_{n+1}^{(0)} = 0, \qquad c_n^{(0)} = 0, \tag{14}$$

and the recursion relation

$$\begin{aligned} &\left(a_{n+1}^{(m+1)} - a_n^{(m+1)}\right) = c_{n+1}^{(m+1)} - r_n b_n^{(m+1)} - s_n\left(a_{n+1}^{(m)} - a_n^{(m)}\right), \qquad m \geqslant 0, \\ &b_{n+1}^{(m+1)} = -s_n b_{n+1}^{(m)} - \left(a_n^{(m)} + a_{n+1}^{(m)}\right), \qquad m \geqslant 0, \\ &c_n^{(m)} = r_n b_{n+1}^{(m)}, \qquad m \geqslant 0. \end{aligned} \tag{15}$$

In equation (15), we take the initial values

$$a_n^{(0)} = -\tfrac{1}{2}, \qquad b_n^{(0)} = 0.$$

Then, the recursion relation (15) uniquely determines $a_n^{(m)}, b_n^{(m)}, c_n^{(m)}, m \geqslant 1$ [10]. For instance, we have

$$\begin{aligned} &a_n^{(1)} = r_n, \quad b_n^{(1)} = 1, \qquad c_n^{(1)} = r_n, \qquad a_n^{(2)} = -r_n r_{n+1} - r_n r_{n-1} - r_n^2 - r_n s_n - r_n s_{n-1}, \\ &b_n^{(2)} = -r_n - r_{n-1} - s_{n-1}, \qquad c_n^{(2)} = -r_n s_n - r_n^2 - r_n r_{n+1}, \ldots. \end{aligned}$$

Now let us introduce the temporal eigenvalue problems

$$\varphi_{nt_m} = V_n^{(m)}\varphi_n = \begin{pmatrix} \sum_{i=0}^{m} a_n^{(i)}\lambda^{2m-2i} + b_n^{(m+1)} & \sum_{i=0}^{m} b_n^{(i)}\lambda^{2m-2i+1} \\ \sum_{i=0}^{m} c_n^{(i)}\lambda^{2m-2i+1} & -\sum_{i=0}^{m} a_n^{(i)}\lambda^{2m-2i} \end{pmatrix}\varphi_n, \qquad m \geqslant 0. \tag{16}$$

The corresponding discrete zero-curvature equations (i.e. the compatibility conditions of equations (9) and (16)) are

$$Y_{nt_m} = \left(EV_n^{(m)}\right)Y_n - Y_n V_n^{(m)}, \qquad m \geqslant 0, \tag{17}$$

which give rise to the hierarchy of the relativistic Toda lattice equations in polynomial form:

$$\begin{cases} r_{nt_m} = c_n^{(m+1)} - r_n b_n^{(m+1)}, & m \geqslant 0, \\ s_{nt_n} = s_n\left(a_n^{(m)} - a_{n+1}^{(m)}\right), & m \geqslant 0. \end{cases} \tag{18}$$

In particular, when $m = 1$, equation (18) becomes the relativistic Toda lattice equations in polynomial form, i.e. equation (6).

Next, we are going to derive the corresponding rational-type integrable lattice hierarchy from the same eigenvalue problem (9). We solve

$$(EY_n)\widehat{\Pi}_n - \widehat{\Pi}_n Y_n = 0, \tag{19}$$

where

$$\widehat{\Pi}_n = \begin{pmatrix} \widehat{a}_n & \widehat{b}_n \\ \widehat{c}_n & -\widehat{a}_n \end{pmatrix}$$

with $\widehat{a}_n = \sum_{m=0}^{\infty} \widehat{a}_n^{(m)}\lambda^{2m}$, $\widehat{b}_n = \sum_{m=0}^{\infty} \widehat{b}_n^{(m)}\lambda^{2m-1}$ and $\widehat{c}_n = \sum_{m=0}^{\infty} \widehat{c}_n^{(m)}\lambda^{2m-1}$. From equation (19), we have the initial relation

$$\widehat{a}_{n+1}^{(0)} - \widehat{a}_n^{(0)} = 0, \qquad \widehat{b}_{n+1}^{(0)} = 0, \qquad \widehat{c}_n^{(0)} = 0, \tag{20}$$

and the recursion relation

$$\begin{aligned} &s_n\left(\widehat{a}_{n+1}^{(m+1)} - \widehat{a}_n^{(m+1)}\right) = \widehat{c}_{n+1}^{(m+1)} - r_n\widehat{b}_n^{(m+1)} - \left(a_{n+1}^{(m)} - a_n^{(m)}\right), \qquad m \geqslant 0, \\ &s_n\widehat{b}_{n+1}^{(m+1)} = -b_{n+1}^{(m)} - \left(a_n^{(m)} + a_{n+1}^{(m)}\right), \qquad m \geqslant 0, \\ &c_n^{(m)} = r_n b_{n+1}^{(m)}, \qquad m \geqslant 0. \end{aligned} \tag{21}$$

We choose the initial values

$$\widehat{a}_n^{(0)} = -\tfrac{1}{2}, \qquad \widehat{b}_n^{(0)} = 0.$$

Then, equation (19) has a unique solution $\widehat{\Pi}_n$ determined by equations (20) and (21). The first few quantities are given as follows:

$$\widehat{a}_n^{(1)} = \frac{r_n}{s_{n-1}s_n}, \qquad \widehat{b}_n^{(1)} = \frac{1}{s_{n-1}}, \qquad \widehat{c}_n^{(1)} = \frac{r_n}{s_n}, \ldots .$$

We introduce following auxiliary eigenvalue problems associated with the eigenvalue problem (9):

$$\varphi_{nt_m} = \widehat{V}_n^{(m)}\varphi_n = \begin{pmatrix} \sum_{i=0}^{m} \widehat{a}_n^{(i)}\lambda^{-2m+2i} + s_{n-1}b_n^{(m+1)} + \widehat{a}_{n-1}^{(m)} & \sum_{i=0}^{m} \widehat{b}_n^{(i)}\lambda^{-2m+2i-1} \\ \sum_{i=0}^{m} \widehat{c}_n^{(i)}\lambda^{-2m+2i-1} & -\sum_{i=0}^{m} \widehat{a}_n^{(i)}\lambda^{-2m+2i} + \widehat{a}_n^{(m)} \end{pmatrix}\varphi_n,$$

$$m \geqslant 0. \tag{22}$$

At this time, the compatibility conditions of equations (9) and (22) read

$$Y_{nt_m} = \left(E\widehat{V}_n^{(m)}\right)Y_n - Y_n\widehat{V}_n^{(m)}, \qquad m \geqslant 0. \tag{23}$$

Equation (23) yields the hierarchy of the relativistic Toda lattice equations in rational form:

$$\begin{cases} r_{t_m} = r_n\widehat{b}_n^{(m)} - \widehat{c}_n^{(m)}, & m \geqslant 0, \\ s_{nt_m} = s_n\left(\widehat{a}_{n+1}^{(m)} - \widehat{a}_n^{(m)}\right), & m \geqslant 0. \end{cases} \tag{24}$$

Following [10], we have the following conception.

Definition 1. *Assume that the lattice hierarchy (1) has the Lax pair (3) and (4). If the Lax operators $V_n^{(m)}$, $m \geqslant 0$, only include the positive powers of the eigenvalue λ, the lattice hierarchy (1) is called the positive integrable hierarchy. In contrast, if the Lax operators $V_n^{(m)}$, $m \geqslant 0$, only include the negative powers of the eigenvalue λ, the lattice hierarchy (1) is called the negative integrable hierarchy.*

According to the above definition, we know that the hierarchy (18) of the relativistic Toda lattice equations in polynomial form is a positive integrable hierarchy, and the hierarchy (24) of the relativistic Toda lattice equations in rational form is a negative integrable hierarchy.

3. The positive integrable coupling hierarchy

In this section, we shall deduce the hierarchy of positive integrable lattice systems from the eigenvalue problem (10). To this end, we first solve the following stationary discrete zero-curvature equation:

$$(E\Gamma_n)U_n - U_n\Gamma_n = \Gamma_{n+1}U_n - U_n\Gamma_n = 0, \tag{25}$$

with

$$\Gamma_n = \begin{pmatrix} a_n & b_n & e_n & f_n \\ c_n & -a_n & g_n & -e_n \\ 0 & 0 & a_n & b_n \\ 0 & 0 & c_n & -a_n \end{pmatrix}.$$

Equation (25) implies

$$\begin{aligned}
&(a_{n+1} - a_n)\lambda^2 + s_n(a_{n+1} - a_n) - c_{n+1}\lambda + r_n b_n\lambda = 0\\
&b_{n+1}\lambda^2 = -s_n b_{n+1} - (a_n + a_{n+1})\lambda,\\
&c_n\lambda^2 = -s_n c_n - r_n(a_n + a_{n+1})\lambda,\\
&(e_{n+1} - e_n)\lambda^2 = g_{n+1}\lambda - r_n f_n\lambda + s_n(e_n - e_{n+1}) - v_n b_n\lambda + w_n(a_n - a_{n+1}),\\
&f_{n+1}\lambda^2 = -s_n f_{n+1} - b_{n+1}w_n - (e_n + e_{n+1})\lambda,\\
&g_n\lambda^2 = -s_n g_n - r_n(e_n + e_{n+1})\lambda - v_n(a_n + a_{n+1})\lambda - w_n c_n.
\end{aligned} \tag{26}$$

First of all, we discuss the locality of the solution of equation (25).

Proposition 1. *If Γ_n solves equation (25), then $\left(E\left(\Gamma_n^2\right)\right) - \Gamma_n^2 = 0$.*

Proof. *Because U_n is invertible, then, from equation (25), we have*

$$(E\Gamma_n) = U_n\Gamma_n U_n^{-1}.$$

Further

$$\left(E\left(\Gamma_n^2\right) = U_n\Gamma_n^2 U_n^{-1}.\right.$$

Hence,

$$\left(E\left(\Gamma_n^2\right) - \Gamma_n^2 = U_n\Gamma_n^2 U_n^{-1} - \Gamma_n^2.\right.$$

A direct calculation shows

$$U_n\Gamma_n^2 = \Gamma_n^2 U_n.$$

So, we have

$$E\left(\Gamma_n^2\right) - \Gamma_n^2 = D\left(\Gamma_n^2\right) = 0.$$

□

From the substitution of

$$a_n=\sum_{m=0}^{\infty}a_n^{(m)}\lambda^{-2m},\qquad b_n=\sum_{m=0}^{\infty}b_n^{(m)}\lambda^{-2m+1},\qquad c_n=\sum_{m=0}^{\infty}c_n^{(m)}\lambda^{-2m+1},$$
$$e_n=\sum_{m=0}^{\infty}e_n^{(m)}\lambda^{-2m},\qquad f_n=\sum_{m=0}^{\infty}f_n^{(m)}\lambda^{-2m+1},\qquad g_n=\sum_{m=0}^{\infty}g_n^{(m)}\lambda^{-2m+1},$$

into equations (26), we can get the initial requirement

$$\begin{aligned}&\left(a_{n+1}^{(0)}-a_n^{(0)}\right)=c_{n+1}^{(0)}-r_nb_n^{(0)},\qquad b_{n+1}^{(0)}=0,\qquad c_n^{(0)}=0,\\&\left(e_{n+1}^{(0)}-e_n^{(0)}\right)=g_{n+1}^{(0)}-r_nf_n^{(0)},\qquad f_{n+1}^{(0)}=0,\qquad g_n^{(0)}=0,\end{aligned}\tag{27}$$

and the recursion relation

$$\begin{aligned}&\left(a_{n+1}^{(m+1)}-a_n^{(m+1)}\right)=c_{n+1}^{(m+1)}-r_nb_n^{(m+1)}+s_n\left(a_n^{(m)}-a_{n+1}^{(m)}\right),\qquad m\geqslant 0,\\&b_{n+1}^{(m+1)}=-s_nb_{n+1}^{(m)}-\left(a_n^{(m)}+a_{n+1}^{(m)}\right),\qquad m\geqslant 0,\\&c_n^{(m+1)}=-s_nc_n^{(m)}-r_n\left(a_n^{(m)}+a_{n+1}^{(m)}\right),\qquad m\geqslant 0,\\&\left(e_{n+1}^{(m+1)}-e_n^{(m+1)}\right)=g_{n+1}^{(m+1)}-r_nf_n^{(m+1)}+s_n\left(e_n^{(m)}-e_{n+1}^{(m)}\right)-v_nb_n^{(m+1)}+w_n\left(a_n^{(m)}-a_{n+1}^{(m)}\right),\\&\qquad m\geqslant 0,\\&f_{n+1}^{(m+1)}=-s_nf_{n+1}^{(m)}-w_nb_{n+1}^{(m)}-\left(e_n^{(m)}+e_{n+1}^{(m)}\right),\qquad m\geqslant 0,\\&g_n^{(m+1)}=-s_ng_n^{(m)}-r_n\left(e_n^{(m)}+e_{n+1}^{(m)}\right)-v_n\left(a_n^{(m)}+a_{n+1}^{(m)}\right)-w_nc_n^{(m)},\qquad m\geqslant 0.\end{aligned}\tag{28}$$

We choose the initial data

$$a_n^{(0)}=-\tfrac{1}{2},\qquad b_n^{(0)}=0,\qquad e_n^{(0)}=-\tfrac{1}{2},\qquad f_n^{(0)}=0.$$

From equation (28), we see that $b_n^{(m+1)}$ and $c_n^{(m+1)}$ can be determined locally by $a_n^{(m)}$, $b_n^{(m)}$ and $c_n^{(m)}$, $m\geqslant 0$, and $f_n^{(m+1)}$ and $g_n^{(m+1)}$ can be determined locally by $a_n^{(m)}$, $b_n^{(m)}$, $c_n^{(m)}$, $e_n^{(m)}$, $f_n^{(m)}$, $g_n^{(m)}$, $m\geqslant 0$. In order to obtain $a_n^{(m+1)}$ and $e_n^{(m+1)}$ from the first and fourth equations in equation (28), we need to use operator $D^{-1}=(E-1)^{-1}$ to solve the relevant difference equations. In what follows, we will show that $a_n^{(m+1)}$ and $e_n^{(m+1)}$ may be deduced by an algebraical method rather than by solving the difference equations. The fact

$$\Gamma_n^2=\begin{pmatrix}a_n^2+b_nc_n & 0 & 2a_ne_n+c_nf_n+b_ng_n & 0\\ 0 & a_n^2+b_nc_n & 0 & 2a_ne_n+c_nf_n+b_ng_n\\ 0 & 0 & a_n^2+b_nc_n & 0\\ 0 & 0 & 0 & a_n^2+b_nc_n\end{pmatrix},$$

and proposition 1 yield

$$a_n^2+b_nc_n=\gamma_1(t),\qquad 2a_ne_n+c_nf_n+b_ng_n=\gamma_2(t),\tag{29}$$

with $\gamma_1(t)$ and $\gamma_2(t)$ being the arbitrary functions of time variable t only. We then obtain two recursion relations for $a_n^{(m+1)}$ and $e_n^{(m+1)}$:

$$\begin{aligned}&a_n^{(m+1)}=\sum_{i=1}^{m}a_n^{(i)}a_n^{(m-i+1)}+\sum_{i=1}^{m+1}b_n^{(i)}c_n^{(m-i+2)}-\gamma_1(t),\qquad m\geqslant 1,\\&e_n^{(m+1)}=\sum_{i=1}^{m}a_n^{(i)}e_n^{(m-i+1)}+\sum_{i=1}^{m+1}c_n^{(i)}f_n^{(m-i+2)}+\sum_{i=1}^{m+1}b_n^{(i)}g_n^{(m-i+2)}-a_n^{(m+1)}-\gamma_2(t),\qquad m\geqslant 1.\end{aligned}\tag{30}$$

Further, we select $\gamma_1(t)=\gamma_2(t)=0$. According to the recursion relations (28) and (30), and through the mathematical induction, we obtain that $a_n^{(m)}$, $b_n^{(m)}$, $c_n^{(m)}$, $e_n^{(m)}$, $f_n^{(m)}$, $g_n^{(m)}$, $m\geqslant 1$, are all local. To sum up the above statement, it turns out that all lattice functions $a_n^{(m)}$, $b_n^{(m)}$, $c_n^{(m)}$,

$e_n^{(m)}$, $f_n^{(m)}$, $g_n^{(m)}$, $m \geqslant 1$, are difference polynomials in the dependent variables r_n, s_n, v_n and w_n. The first few quantities are given by

$$a_n^{(1)} = r_n, \quad b_n^{(1)} = 1, \quad c_n^{(1)} = r_n, \quad e_n^{(1)} = r_n + v_n, \quad f_n^{(1)} = 1, \quad g_n^{(1)} = r_n + v_n,$$
$$a_n^{(2)} = -r_n^2 - r_n s_n - r_n r_{n+1} - r_n r_{n-1} - r_n s_{n-1}, \qquad b_n^{(2)} = -r_{n-1} - r_n - s_{n-1},$$
$$c_n^{(2)} = -r_n^2 - r_n s_n - r_n r_{n+1},$$
$$e_n^{(2)} = -r_n^2 - r_n r_{n+1} - r_n r_{n-1} - r_n(2v_n - v_{n+1} - w_n - w_{n-1} - s_{n-1} - v_{n-1}) - r_{n-1}v_n - s_n v_n - r_{n+1}v_n - v_n s_{n-1},$$
$$f_n^{(2)} = -r_{n-1} - r_n - v_{n-1} - v_n - s_{n-1} - w_{n-1},$$
$$g_n^{(2)} = -r_n(r_n + s_n + r_{n+1} + 2v_n + w_n + v_{n+1}) - v_n s_n - r_{n+1}v_n, \ldots.$$

Let us set

$$W_n^{(m)} \equiv \sum_{i=0}^{m} \begin{pmatrix} a_n^{(i)} & b_n^{(i)}\lambda & e_n^{(i)} & f_n^{(i)}\lambda \\ c_n^{(i)}\lambda & -a_n^{(i)} & g_n^{(i)}\lambda & -e_n^{(i)} \\ 0 & 0 & a_n^{(i)} & b_n^{(i)}\lambda \\ 0 & 0 & c_n^{(i)}\lambda & -a_n^{(i)} \end{pmatrix} \lambda^{2m-2i}, \qquad m \geqslant 0, \tag{31}$$

and take a modification

$$\Delta_n^{(m)} = \begin{pmatrix} b_n^{(m+1)} & 0 & f_n^{(m+1)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_n^{(m+1)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then we define the auxiliary Lax operators

$$W_n^{[m]} = W_n^{(m)} + \Delta_n^{(m)}, \qquad m \geqslant 0. \tag{32}$$

For all $m \geqslant 0$, we introduce the following auxiliary eigenvalue problems:

$$\varphi_{n_{t_m}} = W_n^{[m]}\varphi_n, \qquad m \geqslant 0. \tag{33}$$

Then the compatibility conditions of equations (10) and (33) are

$$U_{n_{t_m}} = \left(EW_n^{[m]}\right)U_n - U_n W_n^{[m]}, \qquad m \geqslant 0, \tag{34}$$

which give rise to the following hierarchy of integrable lattice equations:

$$r_{nt_m} = c_n^{(m+1)} - r_n b_n^{(m+1)}, \qquad m \geqslant 0, \tag{35a}$$

$$s_{nt_m} = s_n\left(a_n^{(m)} - a_{n+1}^{(m)}\right), \qquad m \geqslant 0, \tag{35b}$$

$$v_{nt} = g_n^{(m+1)} - r_n f_n^{(m+1)} - v_n b_n^{(m+1)}, \qquad m \geqslant 0, \tag{35c}$$

$$w_{nt_m} = s_n\left(e_n^{(m)} - e_{n+1}^{(m)}\right) + w_n\left(a_n^{(m)} - a_{n+1}^{(m)}\right), \qquad m \geqslant 0. \tag{35d}$$

It is easy to find that equations (35*a*) and (35*b*) constitute the hierarchy (18) of the relativistic Toda lattice equations in polynomial form, and the Lax operators (32) only involve positive powers of the eigenvalue λ. Therefore, (35) is a positive integrable coupling hierarchy associated with the eigenvalue problem (10). It is easy to see that the first nonlinear lattice equation in (35), when $m = 1$, under $t_1 - > t$, is

$$\begin{cases} r_{nt_1} = r_n(r_{n-1} - r_{n+1}) + r_n(s_{n-1} - s_n), \\ s_{nt_1} = s_n(r_n - r_{n+1}), \\ v_{nt_1} = r_n(s_{n-1} - s_n + w_{n-1} - w_n + r_{n-1} - r_{n+1} + v_{n-1} - v_{n+1}) \\ \qquad + v_n(r_{n-1} - r_{n+1} + s_{n-1} - s_n), \\ w_{nt_1} = s_n(r_n + v_n - r_{n+1} - v_{n+1}) + w_n(r_n - r_{n+1}). \end{cases} \tag{36}$$

Clearly, equation (36) is an integrable coupling of the relativistic Toda lattice model (6) in polynomial form.

4. The negative integrable coupling hierarchy

Now, we would like to derive the negative integrable lattice hierarchy associated with the eigenvalue problem (10). To this end, we consider another stationary discrete zero-curvature equation

$$(E\widehat{\Gamma}_n)U_n - U_n\widehat{\Gamma} = 0, \tag{37}$$

where

$$\widehat{\Gamma}_n = \sum_{m=0}^{\infty}\begin{pmatrix} A_n^{(m)}\lambda & B_n^{(m)} & H_n^{(m)}\lambda & F_n^{(m)} \\ C_n^{(m)} & -A_n^{(m)}\lambda & G_n^{(m)} & -H_n^{(m)}\lambda \\ 0 & 0 & A_n^{(m)}\lambda & B_n^{(m)} \\ 0 & 0 & C_n^{(m)} & -A_n^{(m)}\lambda \end{pmatrix}\lambda^{2m-1}.$$

Solving (37), we have the initial relation

$$\begin{aligned}
&s_n\left(A_{n+1}^{(0)} - A_n^{(0)}\right) = r_nB_n^{(0)} - C_{n+1}^{(0)},\ s_nB_{n+1}^{(0)} = 0, \qquad C_n^{(0)} = r_nB_{n+1}^{(0)},\\
&s_n\left(H_{n+1}^{(0)} - H_n^{(0)}\right) = -v_nB_{n+1}^{(0)} - r_nF_n^{(0)} + G_n^{(0)} + w_n\left(A_n^{(0)} - A_{n+1}^{(0)}\right),\\
&s_nF_{n+1}^{(0)} = -w_nB_{n+1}^{(0)}, \qquad s_nG_n^{(0)} = -w_nC_n^{(0)},
\end{aligned} \tag{38}$$

and the recursion relation

$$\begin{aligned}
&s_n\left(A_{n+1}^{(m+1)} - A_n^{(m+1)}\right) = r_nB_n^{(m+1)} - C_{n+1}^{(m+1)} + \left(A_n^{(m)} - A_{n+1}^{(m)}\right), \qquad m \geqslant 0,\\
&s_nB_{n+1}^{(m+1)} = -B_{n+1}^{(m)} - \left(A_n^{(m)} + A_{n+1}^{(m)}\right),\\
&s_nC_n^{(m+1)} = -C_n^{(m)} - r_n\left(A_n^{(m)} + A_{n+1}^{(m)}\right),\\
&s_n\left(H_{n+1}^{(m+1)} - H_n^{(m+1)}\right) = -v_nB_n^{(m+1)} - r_nF_n^{(m+1)} + G_{n+1}^{(m+1)} + \left(H_n^{(m)} - H_{n+1}^{(m)}\right)\\
&\qquad + w_n\left(A_n^{(m+1)} - A_{n+1}^{(m+1)}\right), \qquad m \geqslant 0,\\
&s_nF_{n+1}^{(m+1)} = -F_{n+1}^{(m)} - w_nB_{n+1}^{(m+1)} - \left(H_n^{(m)} + H_{n+1}^{(m)}\right), \qquad m \geqslant 0,\\
&s_nG_n^{(m+1)} = -G_n^{(m)} - v_n\left(A_n^{(m)} + A_{n+1}^{(m)}\right) - w_nC_n^{(m+1)} - r_n\left(H_n^{(m)} + H_{n+1}^{(m)}\right), \qquad m \geqslant 0.
\end{aligned} \tag{39}$$

We choose the initial values satisfying the above initial relation (38):

$$A_n^{(0)} = -\tfrac{1}{2}, \qquad B_n^{(0)} = 0, \qquad H_n^{(0)} = -\tfrac{1}{2}, \qquad F_n^{(0)} = 0.$$

From proposition 1, in the same way as for the recursion relations (30), we have

$$\begin{aligned}
&A_n^{(m+1)} = \sum_{i=1}^{m} A_n^{(i)}A_n^{(m+1-i)} + \sum_{i=1}^{m+1} B_n^{(i)}C_n^{(m+2-i)} - \gamma_3(t),\\
&H_n^{(m+1)} = 2\sum_{i=1}^{m} A_n^{(i)}H_n^{(m+1-i)} + \sum_{i=1}^{m+1} C_n^{(i)}F_n^{(m+1-i)} + \sum_{i=1}^{m} B_n^{(i)}G_n^{(m+1-i)} - A_n^{(m+1)} - \gamma_4(t).
\end{aligned} \tag{40}$$

In equation (40), $\gamma_3(t)$ and $\gamma_4(t)$ are arbitrary functions. If $\gamma_3(t) = \gamma_4(t) = 0$ is chosen, then all lattice functions $A_n^{(m)}, B_n^{(m)}, C_n^{(m)}, H_n^{(m)}, F_n^{(m)}, G_n^{(m)}$, $m \geqslant 1$ are uniquely determined by equations (39) and (40). Then, through the mathematical induction, we can obtain that they are all local, and are just rational functions in the dependent variables r_n, s_n, v_n and w_n. In particular, we have

$$\begin{aligned}
&A_n^{(1)} = \frac{r_n}{s_ns_{n-1}}, \qquad B_n^{(1)} = \frac{1}{s_{n-1}}, \qquad C_n^{(1)} = \frac{r_n}{s_n},\\
&H_n^{(1)} = -\frac{r_nw_n}{s_n^2s_{n-1}} + \frac{r_n}{s_ns_{n-1}} + \frac{v_n}{s_ns_{n-1}} - \frac{r_nw_{n-1}}{s_ns_{n-1}},\\
&F_n^{(1)} = \frac{1}{s_{n-1}} - \frac{w_{n-1}}{s_{n-1}^2}, \qquad G_n^{(1)} = \frac{v_n}{s_n} + \frac{r_n}{s_n} - \frac{r_nw_n}{s_n^2}, \ldots.
\end{aligned}$$

Let us write

$$\widehat{W}_n^{(m)} = \sum_{i=0}^{m} \begin{pmatrix} A_n^{(i)}\lambda & B_n^{(i)} & H_n^{(i)}\lambda & F_n^{(i)} \\ C_n^{(i)} & -A_n^{(i)}\lambda & G_n^{(i)} & -H_n^{(i)}\lambda \\ 0 & 0 & A_n^{(i)}\lambda & B_n^{(i)} \\ 0 & 0 & C_n^{(i)} & -A_n^{(i)}\lambda \end{pmatrix} \lambda^{-2m+2i-1}, \qquad m \geqslant 0,$$

and select the modification term

$$\widehat{\Delta}_n^{(m)} = \begin{pmatrix} A_{n-1}^{(m)} - s_{n-1}B_n^{(m+1)} & 0 & -F_n^{(m)} - H_n^{(m)} & 0 \\ 0 & A_n^{(m)} & 0 & H_n^{(m)} \\ 0 & 0 & A_{n-1}^{(m)} - s_{n-1}B_n^{(m+1)} & 0 \\ 0 & 0 & 0 & A_n^{(m)} \end{pmatrix}.$$

Then we introduce auxiliary Lax operators

$$\widehat{W}_n^{[m]} = \widehat{W}_n^{(m)} + \widehat{\Delta}_n^{(m)}, \qquad m \geqslant 0. \tag{41}$$

Through a straightforward calculation, we find that

$$U_{nt_m} = \big(E\widehat{W}_n^{[m]}\big)U_n - U_n\widehat{W}_n^{[m]} \tag{42}$$

implies the hierarchy of integrable lattice systems

$$r_{nt_m} = s_n C_n^{(m+1)} + r_n A_{n+1}^{(m)} - r_n s_{n-1} B_n^{(m+1)} - r_n A_{n-1}^{(m)} = -C_n^{(m)} + r_n B_n^{(m)}, \qquad m \geqslant 0, \tag{43a}$$

$$s_{nt_m} = s_n\big(A_{n+1}^{(m)} - A_n^{(m)}\big) = -\big(A_{n+1}^{(m-1)} - A_n^{(m-1)}\big) - r_n B_n^{(m)} + C_{n+1}^{(m)}, \qquad m \geqslant 0, \tag{43b}$$

$$v_{nt_m} = s_n G_n^{(m+1)} + w_n C_n^{(m+1)} + v_n A_{n+1}^{(m)} - v_n\big(A_{n-1}^{(m)} + s_{n-1}B_n^{(m+1)}\big) + r_n\big(F_n^{(m)} + H_n^{(m)} + H_{n+1}^{(m)}\big), \quad m \geqslant 0, \tag{43c}$$

$$w_{nt_m} = w_n\big(A_{n+1}^{(m)} - A_n^{(m)}\big) + s_n\big(H_{n+1}^{(m)} - H_n^{(m)}\big), \qquad m \geqslant 0. \tag{43d}$$

Obviously, equations (43*a*) and (43*b*) form the hierarchy (24) of the relativistic Toda lattice equations in rational form, and the Lax operators (41) only involve negative powers of the eigenvalue λ. Thus, (43) is a negative integrable coupling hierarchy associated with the eigenvalue problem (10).

It is easy to verify that the first nonlinear lattice system in (43), when $m = 1$, under $t_1 -> t$, is

$$\begin{cases} r_{nt} = \dfrac{r_n}{s_{n-1}} - \dfrac{r_n}{s_n}, \\ s_{nt} = \dfrac{r_{n+1}}{s_{n+1}} - \dfrac{r_n}{s_{n-1}}, \\ v_{nt} = \dfrac{r_n}{s_{n-1}} - \dfrac{r_n}{s_n} + r_n\left(\dfrac{w_n}{s_n^2} - \dfrac{w_{n-1}}{s_{n-1}^2}\right), \\ w_{nt} = \dfrac{r_{n+1}}{s_{n+1}} - \dfrac{r_n}{s_{n-1}} + \dfrac{v_{n+1}}{s_{n+1}} - \dfrac{v_n}{s_{n-1}} + \dfrac{r_n w_{n-1}}{s_{n-1}^2} - \dfrac{r_{n+1}w_{n+1}}{s_{n+1}^2}. \end{cases}$$

According to equation (8), we know that this is an integrable coupling of the relativistic Toda lattice model (7) in rational form.

5. Hamiltonian structures for the integrable coupling hierarchies

In this section, we are going to establish the Hamiltonian structures for the integrable couplings (35) and (43) by means of the discrete variational identity.

First, let us introduce some concepts for further discussion. The variational derivative, the Gateaux derivative and the inner product are defined, respectively, by

$$\frac{\delta \tilde{H}_n}{\delta u_n} = \sum_{m\in Z} E^{-m}\left(\frac{\partial H_n}{\partial u_{n+m}}\right), \qquad J'(u_n)[v_n] = \frac{\partial}{\partial \varepsilon} J(u_n + \varepsilon v_n)|_{\varepsilon=0},$$
$$\langle f_n, g_n\rangle = \sum_{n\in Z} (f_n, g_n)_{R^4},$$

where f_n, g_n are required to rapidly vanish at infinity, and $(f_n, g_n)_{R^4}$ denotes the standard inner product of f_n and g_n in the Euclidean space R^4.

An operator J is called a discrete Hamiltonian operator, if J is a skew-symmetric operator satisfying the Jacobi identity [7, 10], i.e.

$$J = -J^*,$$
$$\langle J'(u_n)[Jf_n]g_n, h_n\rangle + \mathrm{Cycle}(f_n, g_n, h_n) = 0.$$

In what follows, we introduce a non-degenerate symmetric bilinear form and present related discrete variational identity. Let us consider a set of matrices

$$\omega_1 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\omega_3 = \begin{pmatrix} 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \omega_4 = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\omega_5 = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \omega_6 = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\omega_7 = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \omega_8 = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that

$$G = \mathrm{span}\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}, \qquad G_1 = \mathrm{span}\{\omega_1, \omega_2, \omega_3, \omega_4\},$$
$$G_2 = \mathrm{span}\{\omega_5, \omega_6, \omega_7, \omega_8\}$$

construct three Lie algebras. Then, we can obtain three corresponding loop algebras $\tilde{G}, \tilde{G}_1, \tilde{G}_2$. For example, the bases of the loop algebra $\tilde{G}$ are

$$\omega_i(m) = \lambda^m \omega_i, \qquad i = 1, 2, \ldots 8, \qquad m \in Z,$$

and the commutation operation in the $\tilde{G}$ is as follows:

$$[\omega_i(m), \omega_j(n)] = [\omega_i, \omega_j]\lambda^{m+n}, \qquad i = 1, 2, \ldots, 8, n, \qquad m \in Z.$$

It is apparent that $G = G_1 \oplus G_2$, $[G_1, G_2] \equiv G_1G_2 - G_2G_1 \subseteq G_2$. Then we obtain that $\tilde{G}$ is a semi-direct sum of $\tilde{G}_1$ and $\tilde{G}_2$ [21–24]. Obviously, for every $\Omega_n \in \tilde{G}$, Ω_n is of the following form:

$$\Omega_n = \begin{pmatrix} Q_n & Z_n \\ \tilde{0} & Q_n \end{pmatrix},$$

where Q_n, Z_n are 2×2 matrices, $\tilde{0}$ is a 2×2 zero matrix.

Following [24], we define a map

$$\xi : \tilde{G} \to R^8, \qquad A \mapsto (a_1, a_2, \ldots a_8)^T, \qquad A = \begin{pmatrix} a_1 & a_2 & a_5 & a_6 \\ a_3 & a_4 & a_7 & a_8 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & a_3 & a_4 \end{pmatrix}, \qquad A \in \tilde{G}. \tag{44}$$

Set

$$A = a_1\omega_1 + a_2\omega_2 + a_3\omega_3 + a_4\omega_4 + a_5\omega_5 + a_6\omega_6 + a_7\omega_7 + a_8\omega_8 \in \tilde{G},$$
$$B = b_1\omega_1 + b_2\omega_2 + b_3\omega_3 + b_4\omega_4 + b_5\omega_5 + b_6\omega_6 + b_7\omega_7 + b_8\omega_8 \in \tilde{G}.$$

Then

$$\xi(A) = a, \qquad \xi(B) = b, \tag{45}$$

where

$$a = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)^T, \qquad b = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)^T \in R^8.$$

The commutator $[.,.]$ on R^8 can be denoted by

$$[a, b]^T_{R^8} = a^T R(b), \qquad a, b \in R^8, \tag{46}$$

where

$$R(b) = \begin{pmatrix} 0 & b_2 & -b_3 & 0 & 0 & b_6 & -b_7 & 0 \\ b_3 & b_4 - b_1 & 0 & -b_3 & b_7 & b_8 - b_5 & 0 & -b_7 \\ -b_2 & 0 & b_1 - b_4 & b_2 & -b_6 & 0 & b_5 - b_8 & b_6 \\ 0 & -b_2 & b_3 & 0 & 0 & -b_6 & b_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_2 & -b_3 & 0 \\ 0 & 0 & 0 & 0 & b_3 & b_4 - b_1 & 0 & -b_3 \\ 0 & 0 & 0 & 0 & -b_2 & 0 & b_1 - b_4 & b_2 \\ 0 & 0 & 0 & 0 & 0 & -b_2 & b_3 & 0 \end{pmatrix}.$$

We introduce the invertible matrix

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{47}$$

It is easy to verify that F meets

$$F^T = F,$$

and

$$F(R(b))^T = -R(b)F, \qquad \text{for all} \quad b \in R^8.$$

Therefore, we obtain a non-degenerate symmetric bilinear form on R^8:

$$\langle a, b\rangle_{R^8} = a^T F b. \tag{48}$$

Now, a non-degenerate bilinear form on $\tilde{G}$ may be presented by

$$\begin{aligned}\langle A, B\rangle_{\tilde{G}} &= \langle \xi(A), \xi(B)\rangle_{R^8} \\ &= (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)F(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)^T.\end{aligned} \tag{49}$$

As in [24], in virtue of the non-degenerate bilinear form (49), we can obtain the discrete variational identity

$$\frac{\delta}{\delta u_n}\sum_{n\in Z}\left\langle \left(\Gamma_n U_n^{-1}\right), \frac{\partial U_n}{\partial \lambda}\right\rangle_{\tilde{G}} = \lambda^{-\tau}\frac{\partial}{\partial\lambda}\lambda^{\tau}\left\langle \left(\Gamma_n U_n^{-1}\right), \frac{\partial U_n}{\partial u_n}\right\rangle_{\tilde{G}}. \tag{50}$$

where $u_n = (r_n, s_n, v_n, w_n)^T$, τ is a constant to be fixed.

First, we establish the Hamiltonian structure for the positive integrable coupling hierarchy (35). In our case,

$$\left\langle \left(\Gamma_n U_n^{-1}\right), \frac{\partial U_n}{\partial \lambda}\right\rangle_{\tilde{G}} = \frac{r_n g_n\lambda^2 - r_n s_n g_n - v_n c_n \lambda^2 + s_n v_n c_n + r_n w_n}{r_n^2\lambda^2},$$

$$\left\langle \left(\Gamma_n U_n^{-1}\right), \frac{\partial U_n}{\partial r_n}\right\rangle_{\tilde{G}} = \frac{r_n e_n - v_n a_n}{r_n^2}, \qquad \left\langle \left(\Gamma_n U_n^{-1}\right), \frac{\partial U_n}{\partial s_n}\right\rangle_{\tilde{G}} = \frac{r_n g_n - v_n c_n}{r_n^2\lambda},$$

$$\left\langle \left(\Gamma_n U_n^{-1}\right), \frac{\partial U_n}{\partial v_n}\right\rangle_{\tilde{G}} = \frac{a_n}{r_n}, \qquad \left\langle \left(\Gamma_n U_n^{-1}\right), \frac{\partial U_n}{\partial w_n}\right\rangle_{\tilde{G}} = \frac{c_n}{r_n\lambda}.$$

The substitution of

$$a_n = \sum_{m=0}^{\infty} a_n^{(m)}\lambda^{-2m}, \qquad b_n = \sum_{m=0}^{\infty} b_n^{(m)}\lambda^{-2m+1}, \qquad c_n = \sum_{m=0}^{\infty} c_n^{(m)}\lambda^{-2m+1},$$

$$e_n = \sum_{m=0}^{\infty} e_n^{(m)}\lambda^{-2m}, \qquad g_n = \sum_{m=0}^{\infty} g_n^{(m)}\lambda^{-2m+1}$$

into equation (50), and comparing the coefficients of λ^{-2m-1}, we get

$$\begin{pmatrix}\frac{\delta}{\delta r_n}\\ \frac{\delta}{\delta s_n}\\ \frac{\delta}{\delta v_n}\\ \frac{\delta}{\delta w_n}\end{pmatrix}\sum_{n\in Z}\left(\frac{(s_n v_n c_n^{(m)} + r_n w_n c_n^{(m)} + r_n g_n^{(m+1)}}{r_n^2} - \frac{s_n g_n^{(m)} + v_n c_n^{(m+1)}}{r_n^2}\right)$$

$$= (\tau - 2m)\begin{pmatrix}\frac{r_n e_n^{(m)} - v_n a_n^{(m)}}{r_n^2}\\ \frac{r_n g_n^{(m)} - v_n c_n^{(m)}}{r_n^2}\\ \frac{a_n^{(m)}}{r_n}\\ \frac{c_n^{(m)}}{r_n}\end{pmatrix}, \qquad m \geqslant 1. \tag{51}$$

When $m = 0$ in (51), we find that $\tau = 0$. So we obtain

$$\begin{pmatrix} \frac{\delta}{\delta r_n} \\ \frac{\delta}{\delta s_n} \\ \frac{\delta}{\delta v_n} \\ \frac{\delta}{\delta w_n} \end{pmatrix} \sum_{n\in Z}\left(\frac{s_n g_n^{(m)} + v_n c_n^{(m+1)}}{2mr_n^2} - \frac{\left(s_n v_n c_n^{(m)} + r_n w_n c_n^{(m)} + r_n g_n^{(m+1)}\right)}{2mr_n^2}\right)$$

$$= \begin{pmatrix} \frac{r_n e_n^{(m)} - v_n a_n^{(m)}}{r_n^2} \\ \frac{r_n g_n^{(m)} - v_n c_n^{(m)}}{r_n^2} \\ \frac{a_n^{(m)}}{r_n} \\ \frac{c_n^{(m)}}{r_n} \end{pmatrix}, \qquad m \geqslant 1. \tag{52}$$

Set

$$\tilde{H}_n^{(m)} = \sum_{n\in Z}\left(\frac{s_n g_n^{(m)} + v_n c_n^{(m+1)}}{2mr_n^2} - \frac{\left(s_n v_n c_n^{(m)} + r_n w_n c_n^{(m)} + r_n g_n^{(m+1)}\right)}{2mr_n^2}\right), \qquad m \geqslant 1. \tag{53}$$

We get

$$\frac{\delta \tilde{H}_n^{(m)}}{\delta u_n} = \left(\frac{r_n e_n^{(m)} - v_n a_n^{(m)}}{r_n^2}, \frac{r_n g_n^{(m)} - v_n c_n^{(m)}}{r_n^2}, \frac{a_n^{(m)}}{r_n}, \frac{c_n^{(m)}}{r_n}\right)^T, \qquad m \geqslant 1. \tag{54}$$

From equation (35), we have

$$\begin{pmatrix} c_n^{(m+1)} - r_n b_n^{(m+1)} \\ s_n\left(a_n^{(m)} - a_{n+1}^{(m)}\right) \\ g_n^{(m+1)} - r_n f_n^{(m+1)} - v_n b_n^{(m+1)} \\ s_n\left(e_n^{(m)} - e_{n+1}^{(m)}\right) + w_n\left(a_n^{(m)} - a_{n+1}^{(m)}\right) \end{pmatrix} = \Theta \begin{pmatrix} \frac{r_n e_n^{(m+1)} - v_n a_n^{(m+1)}}{r_n^2} \\ \frac{r_n g_n^{(m+1)} - v_n c_n^{(m+1)}}{r_n^2} \\ \frac{a_n^{(m+1)}}{r_n} \\ \frac{c_n^{(m+1)}}{r_n} \end{pmatrix},$$

where

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & r_n(1-E^{-1}) \\ 0 & 0 & -(1-E)r_n & r_nE^{-1} - Er_n \\ 0 & r_n(1-E^{-1}) & 0 & v_n(1-E^{-1}) \\ -(1-E)r_n & r_nE^{-1} - Er_n & -(1-E)v_n & (v_nE^{-1} - Ev_n) \end{pmatrix}. \tag{55}$$

Using (28) and (30), it is found that

$$\begin{pmatrix} \frac{r_n e_n^{(m+1)} - v_n a_n^{(m+1)}}{r_n^2} \\ \frac{r_n g_n^{(m+1)} - v_n c_n^{(m+1)}}{r_n^2} \\ \frac{a_n^{(m+1)}}{r_n} \\ \frac{c_n^{(m+1)}}{r_n} \end{pmatrix} = \Phi \begin{pmatrix} \frac{r_n e_n^{(m)} - v_n a_n^{(m)}}{r_n^2} \\ \frac{r_n g_n^{(m)} - v_n c_n^{(m)}}{r_n^2} \\ \frac{a_n^{(m)}}{r_n} \\ \frac{c_n^{(m)}}{r_n} \end{pmatrix}, \qquad m \geqslant 0,$$

where the operator Φ is given by

$$\Phi = \begin{pmatrix} \Phi_{11} & \frac{1}{r_n}(1-E)^{-1}r_n(1-E^{-1})s_n - s_n & \Phi_{13} & \Phi_{14} \\ -(1+E)r_n & -s_n & -(1+E)v_n & -w_n \\ 0 & 0 & \Phi_{33} & \frac{1}{r_n}(1-E)^{-1}r_n(1-E^{-1})s_n - s_n \\ 0 & 0 & -(1+E)r_n & -s_n \end{pmatrix}, \tag{56}$$

in which

$$\Phi_{11} = \Phi_{33} = -(1+E)r_n + \frac{1}{r_n}(1-E)^{-1}r_n(E-E^{-1})r_n - \frac{1}{r_n}(1-E)^{-1}s_n(1-E)r_n.$$

$$\begin{aligned}\Phi_{13} = &-\frac{v_n}{r_n^2}(1-E^{-1})r_n(E-E^{-1})r_n + (1-E^{-1})^{-1}\frac{v_n}{r_n}(E-E^{-1})r_n\\ &+\frac{v_n}{r_n}(1-E)^{-1}s_n(1-E)r_n - (1+E)v_n - (1-E^{-1})^{-1}\frac{v_n}{r_n}(E-E^{-1})r_n\\ &+\frac{1}{r_n}(1-E)^{-1}v_n(E-E^{-1})r_n + \frac{1}{r_n}(1-E)^{-1}r_n(E-E^{-1})v_n\\ &-\frac{1}{r_n}(1-E)^{-1}w_n(1-E)r_n - \frac{1}{r_n}(1-E)^{-1}s_n(1-E)v_n.\end{aligned}$$

$$\begin{aligned}\Phi_{14} = &-\frac{v_n}{r_n^2}(1-E)^{-1}r_n(1-E^{-1})s_n - w_n + \frac{1}{r_n}(1-E)^{-1}r_n(1-E^{-1})w_n\\ &+\frac{1}{r_n}(1-E)^{-1}v_n(1-E^{-1})s_n.\end{aligned}$$

Set

$$\chi = \Theta\Phi.$$

It is easy to verify that

$$\chi = \begin{pmatrix} 0 & 0 & -r_n(E-E^{-1})r_n & -r_n(1-E^{-1})s_n\\ 0 & 0 & s_n(1-E)r_n & 0\\ -r_n(E-E^{-1})r_n & -r_n(1-E^{-1})s_n & \chi_{33} & \chi_{34}\\ s_n(1-E)r_n & 0 & \chi_{43} & 0\end{pmatrix}, \qquad (57)$$

where

$$\chi_{33} = -r_n(E-E^{-1})v_n - v_n(E-E^{-1})r_n, \qquad \chi_{34} = -r_n(1-E^{-1})w_n - v_n(1-E^{-1})s_n,$$
$$\chi_{43} = w_n(1-E)r_n + s_n(1-E)v_n.$$

Proposition 2. *The operator Θ in equation (55) is a discrete Hamiltonian operator.*

Proof. *Obviously, the operator Θ is a skew-symmetric operator, i.e.*

$$\Theta = -\Theta^*.$$

Moreover, we can prove that the operator Θ satisfies the Jacobi identity

$$\langle\Theta'(u_n)[\Theta f_n]g_n, h_n\rangle + \mathrm{Cycle}(f_n, g_n, h_n) = 0.$$

The concrete check is given in the appendix. □

Furthermore, we have following assertions.

Proposition 3. *The operator χ is also a discrete Hamiltonian operator. The operators Θ and χ constitute a pair of discrete Hamiltonian operators.*

Proof. *From a direct but complicated verification, we can obtain that:*

(i) the operator χ is a skew-symmetric operator and satisfies the Jacobi identity;
(ii) for an arbitrary real number α, β, $\alpha\Theta + \beta\chi$ is still a discrete Hamiltonian operator. Namely, the operators Θ and χ are compatible.

Therefore, the operators Θ and χ constitute a pair of discrete Hamiltonian operators. □

At this time, we can rewrite the integrable coupling systems (35) in the following Hamiltonian forms:

$$u_{n_{t_m}} = \begin{pmatrix} r_n \\ s_n \\ v_n \\ w_n \end{pmatrix}_{t_m} = \Theta \frac{\delta \tilde{H}_n^{(m)}}{\delta u_n} = \Theta \begin{pmatrix} \frac{r_n e_n^{(m+1)} - v_n a_n^{(m+1)}}{r_n^2} \\ \frac{r_n g_n^{(m+1)} - v_n c_n^{(m+1)}}{r_n^2} \\ \frac{a_n^{(m+1)}}{r_n} \\ \frac{c_n^{(m+1)}}{r_n} \end{pmatrix} = \Theta \Phi^{m-1} \frac{\delta \tilde{H}_n^{(0)}}{\delta u_n}, \qquad m > 0. \tag{58}$$

Further, we can show that the integrable coupling systems (35) have discrete bi-Hamiltonian structures [7, 11]

$$u_{n_{t_m}} = \begin{pmatrix} r_n \\ s_n \\ v_n \\ w_n \end{pmatrix}_{t_m} = \Theta \frac{\delta \tilde{H}_n^{(m)}}{\delta u_n} = \chi \frac{\delta \tilde{H}_n^{(m)}}{\delta u_n}, \qquad m \geqslant 1. \tag{59}$$

Therefore, equation (35) is a hierarchy of discrete bi-Hamiltonian systems.

Based on a given Hamiltonian operator Θ, we can define a corresponding Poisson bracket [11, 12]

$$\{f_n, g_n\}_\Theta = \left\langle \frac{\delta f_n}{\delta u_n}, \Theta \frac{\delta g_n}{\delta u_n} \right\rangle = \sum_{n \in Z} \left(\frac{\delta f_n}{\delta u_n}, \Theta \frac{\delta g_n}{\delta u_n} \right)_{R^4}. \tag{60}$$

Proposition 4. $\{\tilde{H}_m\}_{m \geqslant 1}$ *defined by* (53) *forms an infinite set of conserved functionals of the integrable coupling hierarchy* (35), *and* $\tilde{H}_m$, $m \geqslant 1$, *are in involution in pairs with respect to the Poisson* (60).

Proof. *We can find that*

$$\chi^* = -\chi.$$

Namely

$$(\Theta\Phi)^* = -(\Theta\Phi),$$

and then

$$\Phi^*\Theta = \Theta\Phi.$$

Therefore, we have

$$\begin{aligned} \left\{\tilde{H}_n^{(m)}, \tilde{H}_n^{(l)}\right\}_\Theta &= \left\langle \frac{\delta \tilde{H}_n^{(m)}}{\delta u_n}, \Theta \frac{\delta \tilde{H}_n^{(m)}}{\delta u_n} \right\rangle = \left\langle \Phi^{m-1} \frac{\delta \tilde{H}_n^{(1)}}{\delta u_n}, \Theta \Phi^{l-1} \frac{\delta \tilde{H}_n^{(1)}}{\delta u_n} \right\rangle \\ &= \left\langle \Phi^{m-1} \frac{\delta \tilde{H}_n^{(1)}}{\delta u_n}, \Phi * \Theta \Phi^{l-2} \frac{\delta \tilde{H}_n^{(1)}}{\delta u_n} \right\rangle \\ &= \left\langle \Phi^{m} \frac{\delta \tilde{H}_n^{(1)}}{\delta u_n}, \Theta \Phi^{l-2} \frac{\delta \tilde{H}_n^{(1)}}{\delta u_n} \right\rangle = \left\{\tilde{H}_n^{(m+1)}, \tilde{H}_n^{(l-1)}\right\}_\Theta \\ &= \cdots = \left\{\tilde{H}_n^{(m+l-1)}, \tilde{H}_n^{(1)}\right\}_\Theta, \qquad m, \ l \geqslant 1. \end{aligned}$$

Analogously, we can obtain

$$\left\{\tilde{H}_n^{(l)}, \tilde{H}_n^{(m)}\right\}_\Theta = \left\{\tilde{H}_n^{(m+l-1)}, \tilde{H}_n^{(1)}\right\}_\Theta, \qquad m, \ l \geqslant 1.$$

Then, we have

$$\left\{\tilde{H}_n^{(m)}, \tilde{H}_n^{(l)}\right\}_\Theta = 0, \qquad m,\ l \geqslant 1, \tag{61}$$

and

$$\begin{aligned}\left(\tilde{H}_n^{(m)}\right)_{t_l} &= \left(\sum_{n\in Z} H_n^{(m)}\right)_{t_l} = \left\langle \frac{\delta \tilde{H}_n^{(m)}}{\delta u_n}, u_{nt_l}\right\rangle = \left\langle \frac{\delta \tilde{H}_n^{(m)}}{\delta u_n}, \Theta \frac{\delta \tilde{H}_n^{(l)}}{\delta u_n}\right\rangle \\ &= \left\{\tilde{H}_n^{(m)}, \tilde{H}_n^{(l)}\right\}_\Theta = 0, \qquad m,\ l \geqslant 1.\end{aligned} \tag{62}$$

In summary, we arrive at the following result. □

Proposition 5. *The integrable couplings in* (35) *or the discrete Hamiltonian equations in* (58) *are all discrete Liouville integrable systems.*

Now we substitute Γ_n in the discrete variational identity (50) with $\widehat{\Gamma}_n$; then it is found that

$$\left\langle (\widehat{\Gamma}_n U_n^{-1}), \frac{\partial U_n}{\partial \lambda}\right\rangle_{\tilde{G}} = \frac{r_n G_n \lambda^2 - r_n s_n G_n - v_n C_n \lambda^2 + s_n v_n C_n + r_n w_n}{r_n^2 \lambda^2},$$

$$\left\langle (\widehat{\Gamma}_n U_n^{-1}), \frac{\partial U_n}{\partial r_n}\right\rangle_{\tilde{G}} = \frac{r_n H_n - v_n A_n}{r_n^2}, \qquad \left\langle (\widehat{\Gamma}_n U_n^{-1}), \frac{\partial U_n}{\partial s_n}\right\rangle_{\tilde{G}} = \frac{r_n G_n - v_n C_n}{r_n^2 \lambda},$$

$$\left\langle (\widehat{\Gamma}_n U_n^{-1}), \frac{\partial U_n}{\partial v_n}\right\rangle_{\tilde{G}} = \frac{A_n}{r_n}, \qquad \left\langle (\widehat{\Gamma}_n U_n^{-1}), \frac{\partial U_n}{\partial w_n}\right\rangle_{\tilde{G}} = \frac{C_n}{r_n \lambda}.$$

Substituting expansions

$$A_n = \sum_{m=0}^{\infty} A_n^{(m)} \lambda^{2m}, \qquad B_n = \sum_{m=0}^{\infty} B_n^{(m)} \lambda^{2m-1}, \qquad C_n = \sum_{m=0}^{\infty} C_n^{(m)} \lambda^{2m-1},$$

$$H_n = \sum_{m=0}^{\infty} H_n^{(m)} \lambda^{2m}, \qquad G_n = \sum_{m=0}^{\infty} G_n^{(m)} \lambda^{2m-1}$$

into the equation

$$\frac{\delta}{\delta u_n} \sum_{n\in Z} \left\langle (\widehat{\Gamma}_n U_n^{-1}), \frac{\partial U_n}{\partial \lambda}\right\rangle_{\tilde{G}} = \lambda^{-\tau} \frac{\partial}{\partial \lambda} \lambda^{\tau} \left\langle (\widehat{\Gamma}_n U_n^{-1}), \frac{\partial U_n}{\partial u_n}\right\rangle_{\tilde{G}}$$

we have

$$\frac{\delta \tilde{G}_n^{(m)}}{\delta u_n} = \left(\frac{r_n H_n^{(m)} - v_n A_n^{(m)}}{r_n^2}, \frac{r_n G_n^{(m+1)} - v_n C_n^{(m+1)}}{r_n^2}, \frac{A_n^{(m)}}{r_n}, \frac{C_n^{(m+1)}}{r_n} \right)^T, \qquad m \geqslant 1, \tag{63}$$

where

$$\tilde{G}_n^{(m)} = \sum_{n\in Z} \left(\frac{r_n G_n^{(m)} + s_n v_n C_n^{(m+1)} + r_n w_n C_n^{(m+1)}}{2m r_n^2} - \frac{r_n s_n G_n^{(m+1)} + v_n C_n^{(m)}}{2m r_n^2} \right).$$

By using (63) and noting (39) and (40), we have

$$\left(\begin{array}{c} s_n C_n^{(m+1)} + r_n A_{n+1}^{(m)} - r_n s_{n-1} B_n^{(m+1)} - r_n A_{n-1}^{(m)} \\ s_n \left(A_{n+1}^{(m)} - A_n^{(m)}\right) \\ s_n G_n^{(m+1)} + w_n C_n^{(m+1)} + v_n A_{n+1}^{(m)} - v_n \left(A_{n-1}^{(m)} + s_{n-1} B_n^{(m+1)}\right) + r_n \left(F_n^{(m)} + H_n^{(m)} + H_{n+1}^{(m)}\right) \\ w_n \left(A_{n+1}^{(m)} - A_n^{(m)}\right) + s_n \left(H_{n+1}^{(m)} - H_n^{(m)}\right) \end{array} \right)$$

$$= \chi \begin{pmatrix} \frac{r_n H_n^{(m)} - v_n A_n^{(m)}}{r_n^2} \\ \frac{r_n G_n^{(m+1)} - v_n C_n^{(m+1)}}{r_n^2} \\ \frac{A_n^{(m)}}{r_n} \\ \frac{C_n^{(m+1)}}{r_n} \end{pmatrix}.$$

Moreover, from (39) and (40), we can find that

$$\begin{pmatrix} \frac{r_n H_n^{(m)} - v_n A_n^{(m)}}{r_n^2} \\ \frac{r_n G_n^{(m+1)} - v_n C_n^{(m+1)}}{r_n^2} \\ \frac{A_n^{(m)}}{r_n} \\ \frac{C_n^{(m+1)}}{r_n} \end{pmatrix} = \Psi \begin{pmatrix} \frac{r_n H_n^{(m-1)} - v_n A_n^{(m-1)}}{r_n^2} \\ \frac{r_n G_n^{(m)} - v_n C_n^{(m)}}{r_n^2} \\ \frac{A_n^{(m-1)}}{r_n} \\ \frac{C_n^{(m)}}{r_n} \end{pmatrix}, \qquad m \geqslant 1,$$

where

$$\Psi = \begin{pmatrix} -\frac{1}{r_n}(1-E)^{-1}\frac{1}{s_n}(1-E)r_n & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \frac{1}{s_n}(1+E)(1-E)^{-1}\frac{1}{s_n}(1-E)r_n & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ 0 & 0 & -\frac{1}{r_n}(1-E)^{-1}\frac{1}{s_n}(1-E)r & \Psi_{34} \\ 0 & 0 & \frac{1}{s_n}(1+E)(1-E)^{-1}\frac{1}{s_n}(1-E)r_n & \Psi_{44} \end{pmatrix},$$

in which

$$\Psi_{12} = \Psi_{34} = \frac{1}{r_n}(1-E)^{-1}\frac{1}{s_n}(r_n E^{-1} - E r_n),$$

$$\Psi_{22} = \Psi_{44} = \frac{1}{s_n} - \frac{1}{s_n}(1+E)(1-E)^{-1}\frac{1}{s_n}(r_n E^{-1} - E r_n),$$

$$\Psi_{13} = r_n(1-E)^{-1}\frac{w_n}{s_n}(1-E)r_n + \frac{v_n}{r_n^2}(1-E)^{-1}\frac{1}{s_n}(1-E)r_n - \frac{1}{r_n}(1-E)^{-1}\frac{1}{s_n}(1-E)v_n,$$

$$\begin{aligned}\Psi_{14} = &-\frac{1}{r_n}(1-E)^{-1}\frac{w_n}{s_n^2}(r_n E^{-1} - E r_n) - \frac{v_n}{r_n^2}(1-E)^{-1}\frac{1}{s_n^2}(r_n E^{-1} - E r_n) \\ &+ \frac{1}{r_n}(1-E)^{-1}\frac{1}{s_n^2}(v_n E^{-1} - E v_n),\end{aligned}$$

$$\begin{aligned}\Psi_{23} = &-\frac{w_n}{s_n^2}(1+E)(1-E)^{-1}\frac{1}{s_n}(1-E)r_n - \frac{1}{s_n}(1+E)(1-E)^{-1}\frac{w_n}{s_n^2}(1-E)r_n \\ &+ \frac{1}{s_n}(1+E)(1-E)^{-1}\frac{1}{s_n}(1-E)v_n,\end{aligned}$$

$$\begin{aligned}\Psi_{24} = &\frac{w_n}{s_n^2} - \frac{w_n}{s_n^2}(1+E)(1-E)^{-1}\frac{r_n}{s_n}(1-E^{-1}) + \frac{w_n}{s_n^2}(1+E)(1-E)^{-1}\frac{1}{s_n}(1-E)r_n \\ &+ \frac{1}{s_n}(1+E)(1-E)^{-1}\frac{r_n w_n}{s_n^2}(1-E^{-1}) \\ &+ \frac{1}{s_n}(1+E)(1-E)^{-1}\frac{w_n}{s_n^2}(1-E)r_n + \frac{1}{s_n}(1+E)(1-E)^{-1}\frac{v_n}{s_n}(1-E^{-1}) \\ &- \frac{1}{s_n}(1+E)(1-E)^{-1}\frac{1}{s_n^2}(1-E)v_n.\end{aligned}$$

It is easy to verify that

$$\Theta = \chi\Psi, \qquad \Psi = \Phi^{-1}.$$

Consequently, we have

$$\begin{pmatrix} s_n C_n^{(m+1)} + r_n A_{n+1}^{(m)} - r_n s_{n-1} B_n^{(m+1)} - r_n A_{n-1}^{(m)} \\ s_n\left(A_{n+1}^{(m)} - A_n^{(m)}\right) \\ s_n G_n^{(m+1)} + w_n C_n^{(m+1)} + v_n A_{n+1}^{(m)} - v_n\left(A_{n-1}^{(m)} + s_{n-1} B_n^{(m+1)}\right) + r_n\left(F_n^{(m)} + H_n^{(m)} + H_{n+1}^{(m)}\right) \\ w_n\left(A_{n+1}^{(m)} - A_n^{(m)}\right) + s_n\left(H_{n+1}^{(m)} - H_n^{(m)}\right) \end{pmatrix}$$

$$= \Theta \begin{pmatrix} \frac{r_n H_n^{(m-1)} - v_n A_n^{(m-1)}}{r_n^2} \\ \frac{r_n G_n^{(m)} - v_n C_n^{(m)}}{r_n^2} \\ \frac{A_n^{(m-1)}}{r_n} \\ \frac{C_n^{(m)}}{r_n} \end{pmatrix}.$$

Then, we can obtain the discrete bi-Hamiltonian forms for equation (43):

$$u_{n_{t_m}} = \begin{pmatrix} r_n \\ s_n \\ v_n \\ w_n \end{pmatrix}_{t_m} = \chi \frac{\delta \tilde{G}_n^{(m)}}{\delta u_n} = \Theta \frac{\delta \tilde{G}_n^{(m-1)}}{\delta u_n}, \qquad m \geqslant 1. \tag{64}$$

And we have the following recursion structures:

$$u_{n_{t_m}} = \begin{pmatrix} r_n \\ s_n \\ v_n \\ w_n \end{pmatrix}_{t_m} = \chi \frac{\delta \tilde{G}_n^{(m)}}{\delta u_n} = \chi \begin{pmatrix} \frac{r_n H_n^{(m)} - v_n A_n^{(m)}}{r_n^2} \\ \frac{r_n G_n^{(m+1)} - v_n C_n^{(m+1)}}{r_n^2} \\ \frac{A_n^{(m)}}{r_n} \\ \frac{C_n^{(m+1)}}{r_n} \end{pmatrix} = \chi \Psi^{m-1} \frac{\delta \tilde{G}_n^{(0)}}{\delta u_n}, \qquad m > 0. \tag{65}$$

Based on proposition 3, we can obtain the following proposition.

Proposition 6. *The integrable couplings in* (43) *have discrete bi-Hamiltonian structures* (64), *and are all Liouville integrable discrete Hamiltonian systems.*

6. Conclusions and remarks

In this paper, firstly, we briefly derived the relativistic Toda lattice hierarchies, both the polynomial form and rational forms. Secondly, we deduced the corresponding positive and negative integrable coupling hierarchies from a four by four matrix eigenvalue problem. Thirdly, we presented a pair of discrete Hamiltonian operators, and established the bi-Hamiltonian structures of the two hierarchies of integrable couplings obtained, by the discrete variational identity, which is derived from a non-degenerate symmetric bilinear form. Finally, Liouville integrability of the two integrable coupling hierarchies obtained is proved.

In addition, we are going to focus on the physical applications of integrable coupling hierarchies (35) and (43), and many other interesting integrability problems, for instance, the inverse scattering transformation [1], the Darboux transformation [15], the symmetries and master symmetries [7, 29] and the symmetry constraint [30], in a further investigation.

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Appendix. The proof of the Jacobi identity in proposition 2

We would like to give a detailed check of the Jacobi identities

$$\langle\Theta'(u_n)[\Theta f_n]g_n, h_n\rangle + \mathrm{Cycle}(f_n, g_n, h_n) = 0,$$

in which the operators Θ are defined by (55).

Assume that

$$f_n = \left(f_n^1, f_n^2, f_n^3, f_n^4\right)^T, \qquad g_n = \left(g_n^1, g_n^2, g_n^3, g_n^4\right)^T, \qquad h_n = \left(h_n^1, h_n^2, h_n^3, h_n^4\right)^T$$

are three arbitrary vector functions, which are required to rapidly vanish at infinity. Here the Gateaux derivative and the inner product are defined, respectively, by

$$\Theta'(u_n)[v_n] = \frac{\partial}{\partial\varepsilon}\Theta(u_n + \varepsilon v_n)|_{\varepsilon=0}, \qquad \langle f_n, g_n\rangle = \sum_{n\in Z}(f_n, g_n)_{R^4},$$

as before. $(f_n, g_n)_{R^4}$ denotes the standard inner product of f_n and g_n in the Euclidean space R^4.

First, we can obtain

$$\begin{aligned}
\langle\Theta'(u_n)[\Theta f_n]g_n, h_n\rangle = \sum_{n\in Z}\Big(&\big(r_n f_n^4 g_n^4 h_n^1 - r_n\big(E^{-1}f_n^4\big)g_n^4 h_n^1 - r_n f_n^4\big(E^{-1}g_n^4\big)h_n^1\\
&+ r_n\big(E^{-1}f_n^4\big)\big(E^{-1}g_n^4\big)h_n^1 - r_n f_n^4 g_n^3 h_n^2 + r_n\big(E^{-1}f_n^4\big)g_n^3h_n^2 + (Er_n)\big(Ef_n^4\big)\big(Eg_n^3\big)h_n^2\\
&- (Er_n)f_n^4\big(Eg_n^3\big)h_n^2 + +r_n f_n^4\big(E^{-1}g_n^4\big)h_n^2 - r_n\big(E^{-1}f_n^4\big)\big(E^{-1}g_n^4\big)h_n^2\\
&- (Er_n)\big(Ef_n^4\big)\big(Eg_n^4\big)h_n^2 + (Er_n)f_n^4\big(Eg_n^4\big)h_n^2 + r_n f_n^4 g_n^2 h_n^3\\
&- r_n\big(E^{-1}f_n^4\big)g_n^2h_n^3 - +r_n f_n^4\big)\big(E^{-1}g_n^2\big)h_n^3 + r_n\big(E^{-1}f_n^4\big)\big(E^{-1}g_n^2\big)h_n^3\\
&+ r_n f_n^2 g_n^4 h_n^3 - r_n\big(E^{-1}f_n^2\big)g_n^4h_n^3 + v_n f_n^4 g_n^4\big)h_n^3 - v_n\big(E^{-1}f_n^4\big)g_n^4h_n^3\\
&- r_n f_n^2\big(E^{-1}g_n^4\big)h_n^3 + r_n\big(E^{-1}f_n^2\big)\big(E^{-1}g_n^4\big)h_n^3 - v_n f_n^4\big(E^{-1}g_n^4\big)h_n^3\\
&+ v_n\big(E^{-1}f_n^4\big)\big(E^{-1}g_n^4\big)h_n^3 - r_n f_n^4 g_n^1 h_n^4 + r_n\big(E^{-1}f_n^4\big)g_n^1h_n^4\\
&+ (Er_n)\big(Ef_n^4\big)\big(Eg_n^1\big)h_n^4 - (Er_n)f_n^4\big(Eg_n^1\big)h_n^4 + r_n f_n^4\big(E^{-1}g_n^2\big)h_n^4\\
&- r_n\big(E^{-1}f_n^4\big)\big(E^{-1}g_n^2\big)h_n^4 - (Er_n)\big(Ef_n^4\big)\big(Eg_n^2\big)h_n^4 + (Er_n)f_n^4\big(Eg_n^2\big)h_n^4\\
&- r_n f_n^2 g_n^3 h_n^4 + r_n\big(E^{-1}f_n^2\big)g_n^3h_n^4 - v_n f_n^4 g_n^3 h_n^4 + v_n\big(E^{-1}f_n^4\big)g_n^3h_n^4\\
&+ (Er_n)\big(Ef_n^2\big)\big(Eg_n^3\big)h_n^4 - (Er_n)f_n^2\big(Eg_n^3\big)h_n^4 + (Ev_n)\big(Ef_n^4\big)\big(Eg_n^3\big)h_n^4\\
&- (Ev_n)f_n^4\big(Eg_n^3\big)h_n^4 + r_n f_n^2\big(E^{-1}g_n^4\big)h_n^4 - r_n\big(E^{-1}f_n^2\big)\big(E^{-1}g_n^4\big)h_n^4\\
&+ v_n f_n^4\big(E^{-1}g_n^4\big)h_n^4 - v_n\big(E^{-1}f_n^4\big)\big(E^{-1}g_n^4\big)h_n^4 - (Er_n)\big(Ef_n^2\big)\big(Eg_n^4\big)h_n^4\\
&+ (Er_n)f_n^2\big(Eg_n^4\big)h_n^4 - (Ev_n)\big(Ef_n^4\big)\big(Eg_n^4\big)h_n^4 + (Ev_n)f_n^4\big(Eg_n^4\big)h_n^4\Big).
\end{aligned}$$

Note that $E^* = E$, through a careful calculation, we obtain that the above sum with a cycle of f_n, g_n and h_n is equal to zero. Therefore, we have

$$\langle\Theta'(u_n)[\Theta f_n]g_n, h_n\rangle + \mathrm{Cycle}(f_n, g_n, h_n) = 0.$$

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